Solutions to the WIKR-06 resit of 11 July 2019

July 5, 2019

Nota bene. These notes briefly indicate solutions to the resit exam, in a manner intellgible to those that have properly studied the course material. It is not necessarily always the case that you will get full marks if you wrote in the exam what I wrote here. The same applies to all other solutions to exams that I've provided.

1a

You can for instance do this using the convolution formula, which applies since X, Y are independent.

$$f_V(v) = \int_{-\infty}^{\infty} f_X(x) f_Y(v-x)\, dx = \int_0^v e^{-x} e^{-(v-x)}\, dx = ve^{-v}.$$

(for $v > 0$. If $v \leq 0$ then obviously $f_V(v) = 0$.)

We recognize the pdf of the Gamma$(2, 1)$.

1b

It suffices to show that $\mathbb{P}(U \leq x) = x$, for every $0 < x < 1$. We have

$$\mathbb{P}(U \leq x) = \mathbb{P}(\frac{X}{X+Y} \leq x) = \mathbb{P}(X \leq x(X+Y)) = \mathbb{P}((1-x)X \leq xY) = \mathbb{P}(X \leq \frac{x}{1-x} Y).$$

Now we can compute

$$\begin{aligned} \mathbb{P}(X \leq \tfrac{x}{1-x} Y) &= \int_0^\infty \int_0^{\frac{x}{1-x}t} e^{-(s+t)}\, ds\, dt = \int_0^\infty e^{-t} \int_0^{\frac{x}{1-x}t} e^{-s}\, ds\, dt = \int_0^\infty e^{-t} \left[1 - e^{-\frac{x}{1-x}t}\right]\, dt \\ &= \int_0^\infty e^{-t}\, dt - \int_0^\infty e^{-t/(1-x)}\, dt = 1 - (1-x) = x. \end{aligned}$$

1c

We compute the joint pdf of (U, V) using the familiar change of variables formula

$$f_{U,V}(u, v) = f_{X,Y}(x(u, v), y(u, v))|J|.$$

Note that X, Y are positive if and only if $0 < U < 1$ and $V > 0$. Also, $X = UV, Y = V - UV = (1-U)V$. Hence,

$$
\begin{aligned}
f_{U,V}(u,v) &= \begin{cases} e^{-v}\left|\det\begin{pmatrix} v & u \\ -v & 1-u \end{pmatrix}\right| & \text{if } 0<u<1, v>0, \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} ve^{-v} & \text{if } 0<u<1, v>0, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

This is the product of f_V with f_U (recall/note f_U is the function which is identically one on $(0,1)$ and zero elsewhere). Hence U, V are *independent*.

2a

The numerator is the number of ways to pick k balls out of the r red balls and $m-k$ balls out of the $n-r$ blue balls. The denominator is the number of ways to pick m balls out of n balls. (Note/recall the answers I give here are not necessarily what you needed to write in order to get full marks.)

2b

By symmetry considerations $\mathbb{E}X_1 = \mathbb{E}X_2 = \cdots = \mathbb{E}X_m$. Also $\mathbb{E}X_1 = \mathbb{P}(X_1 = 1) = \frac{r}{n}$, the fraction of red balls.

$$\mathbb{E}X = \mathbb{E}X_1 + \cdots + \mathbb{E}X_m = m\frac{r}{n}.$$

2c

If $i = j$ then

$$\operatorname{Cov}(X_i, X_j) = \operatorname{Var}(X_i) = \operatorname{Var}(X_1) = \frac{r}{n}(1-\frac{r}{n}) = \frac{r(n-r)}{n^2}.$$

(Note X_1 is $\operatorname{Be}(\frac{r}{n})$.) If $i \neq j$ then

$$
\begin{aligned}
\operatorname{Cov}(X_i, X_i) &= \operatorname{Cov}(X_1, X_2) = \mathbb{E}X_1X_2 - \mathbb{E}X_1\mathbb{E}X_2 = \mathbb{P}(X_1 = X_2 = 1) - \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1) \\
&= \frac{r(r-1)}{n(n-1)} - \frac{r^2}{n^2} = \frac{r(r-n)}{n^2(n-1)}.
\end{aligned}
$$

And

$$
\begin{aligned}
\operatorname{Var}(X) &= \sum_{i=1}^{m} \operatorname{Var}(X_i) + \sum_{\substack{1\le i,j\le m, \\ i\neq j}} \operatorname{Cov}(X_i, X_j) \\
&= m\frac{r(n-r)}{n^2} + m(m-1)\frac{r(r-n)}{n^2(n-1)} = m\cdot\frac{r}{n}\cdot\frac{n-r}{n}\cdot\left(1-\frac{m-1}{n-1}\right) \\
&= m\cdot\frac{r}{n}\cdot\frac{n-r}{n}\cdot\frac{n-m}{n-1}.
\end{aligned}
$$

2d

By definition of convergence in distribution, it suffices to show $\mathbb{P}(Y_n \le x) \to \mathbb{P}(Y \le x)$ for every $x \in \mathbb{R}$. Since $Y \sim \operatorname{Bin}(m,p)$ this follows if $\mathbb{P}(Y_n = k) \to \mathbb{P}(Y = k)$ for $k = 0, \ldots, m$.

Note, for any (fixed) $0 \le k \le m$:

$$\binom{r}{k}/n^k = \frac{\lfloor pn\rfloor(\lfloor pn\rfloor-1)\dots(\lfloor pn\rfloor-k+1)}{k!n^k} = \frac{1}{k!}\left(\frac{\lfloor pn\rfloor}{n}\right)\cdot\dots\cdot\left(\frac{\lfloor pn\rfloor-k+1}{n}\right) \xrightarrow[n\to\infty]{} \frac{1}{k!}p^k.$$

Similarly

$$\binom{n-r}{m-k}/n^{m-k} = \frac{1}{(m-k)!}\left(\frac{n-\lfloor pn\rfloor}{n}\right)\cdot\dots\cdot\left(\frac{n-\lfloor pn\rfloor-m+k+1}{n}\right) \xrightarrow[n\to\infty]{} \frac{1}{(m-k)!}(1-p)^k,$$

and

$$\binom{n}{m}/n^m = \frac{1}{m!}\left(\frac{n}{n}\right)\cdot\dots\cdot\left(\frac{n-m+1}{n}\right) \xrightarrow[n\to\infty]{} \frac{1}{m!}.$$

Combining the three limits we have

$$\frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}} = \frac{\binom{r}{k}\cdot n^{-k}\cdot\binom{n-r}{m-k}\cdot n^{-(m-k)}}{\binom{n}{m}\cdot n^{-m}} \xrightarrow[n\to\infty]{} \frac{m!}{k!(m-k)!}p^k(1-p)^{m-k} = \binom{m}{k}p^k(1-p)^{m-k},$$

which we recognize as the pmf of Y.

3a

See lecture notes.

3b

Same as 3a.

3c

For $-1 \le z \le 1$ we can write $z = \lambda\cdot(-1)+(1-\lambda)\cdot(+1)$ with $\lambda := \left(\frac{1-z}{2}\right)$. (Note $0\le\lambda\le 1$, and $1-\lambda = \left(\frac{1+z}{2}\right)$.) Hence, for φ convex and $-1\le z\le 1$ we have

$$\varphi(z) \le \left(\frac{1-z}{2}\right)\cdot\varphi(-1) + \left(\frac{1+z}{2}\right)\cdot\varphi(1) = \frac{1}{2}\left(\varphi(-1)+\varphi(1)\right) + z\cdot\frac{1}{2}\left(\varphi(1)-\varphi(-1)\right).$$

Hence

$$\mathbb{E}\varphi(Z) \le \frac{1}{2}\left(\varphi(-1)+\varphi(1)\right) + \frac{1}{2}\left(\varphi(1)-\varphi(-1)\right)\cdot\mathbb{E}Z = \frac{1}{2}\left(\varphi(-1)+\varphi(1)\right).$$

Since $z\mapsto e^{tz}$ is convex, we have in particular $\mathbb{E}e^{tZ}\le\cosh(t)$.

3d

We note that, for $t>0$:

$$\mathbb{P}(X-\mathbb{E}X>\lambda) = \mathbb{P}(Y>\lambda) = \mathbb{P}(e^{tY}>e^{t\lambda}) \overset{\text{Markov}}{\le} \frac{\mathbb{E}e^{tY}}{e^{t\lambda}}.$$

3e

Note $Y_1, \dots, Y_n$ (are i.i.d. and) satisfy $\mathbb{E}Y_i = 0, -1 \le Y_i \le 1$. Hence $\mathbb{E}e^{tY_1} = \cdots = \mathbb{E}e^{tY_n} \le \cosh(t) \le e^{t^2/2}$, and also

$$\mathbb{E}e^{tY} \overset{\text{indept.}}{=} \prod_{i=1}^{n} \mathbb{E}e^{tY_i} \le e^{nt^2/2}.$$

Taking $t = \lambda/n$, part (d) gives

$$\mathbb{P}(X - \mathbb{E}X > \lambda) \le e^{-\lambda^2/n} \cdot e^{n(\lambda/n)^2/2} = e^{-\lambda^2/(2n)}.$$

3f

We note that

$$\mathbb{P}(X - \mathbb{E}X < -\lambda) = \mathbb{P}(Y < -\lambda) = \mathbb{P}(-Y > \lambda) = \mathbb{P}((-Y_1) + \cdots + (-Y_n) > \lambda).$$

Now note part (c) also applies to $-Y_1, \dots, -Y_n$. In fact all the previous reasonings apply giving that

$$\mathbb{P}(X - \mathbb{E}X < -\lambda) \le e^{-\lambda^2/(2n)}.$$

Hence $\mathbb{P}(|X - \mathbb{E}X| > \lambda) \le 2e^{-\lambda^2/(2n)}$.